t	3,0	3,1	3,2	3,3	3,4	3,5	3,6	3,8	4,0	4,5
$\Phi_0(t)$	.99865	.99904	.99931	.99952	.99966	.99976	.999841	.999928	.999968	.999997

12.2 Black–Scholes Formula(s) (To Compute Reference Prices)

In a risk-neutral Black–Scholes model, the quoted price of a risky asset is a solution to the *SDE* $dX_t = X_t(rdt + \sigma dW_t)$, $X_0 = x_0 > 0$, where r is the interest rate and $\sigma > 0$ is the volatility and W is a standard Brownian motion. Itô's formula (see Sect. 12.8) yields that

$$X_t^{x_0} = x_0\, e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}, \quad W_t \stackrel{d}{=} \mathcal{N}(0;1).$$

A vanilla (European) payoff of maturity $T > 0$ is of the form $h_T = \varphi(X_T)$. A European option contract written on the payoff h_T is the right to receive h_T at the maturity T. Its price – or premium – at time $t = 0$ is given by $e^{-rt}\mathbb{E}\,\varphi(X_T^{x_0})$ and, more generally at time $t \in [0, T]$, it is given by $e^{-r(T-t)}\mathbb{E}\left(\varphi(X_T^{x_0}) \mid X_t^{x_0}\right) = e^{-r(T-t)}\mathbb{E}\,\varphi(X_{T-t}^{x_0})$. In the case where $\varphi(x) = (x - K)_+$ (*call* with *strike price* K) this premium at time t has a closed form given by

$$\mathrm{Call}_t(x_0, K, R, \sigma, T) = \mathrm{Call}_0(x_0, K, R, \sigma, T - t),$$

where

$$\mathrm{Call}_0(x_0, K, r, \sigma, \tau) = x_0\Phi_0(d_1) - e^{-r\tau}K\,\Phi_0(d_2), \quad \tau > 0, \tag{12.1}$$

with

$$d_1 = \frac{\log\left(\frac{x_0}{K}\right) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}. \tag{12.2}$$

As for the *put* option written on the payoff $h_T = (K - X_T^{x_0})_+$, the premium is

$$\mathrm{Put}_t(x_0, K, r, \sigma, T) = \mathrm{Put}_0(x_0, K, r, \sigma, T - t),$$

where

$$\mathrm{Put}_0(x_0, K, r, \sigma, \tau) = e^{-r\tau}K\,\Phi_0(-d_2) - x_0\Phi_0(-d_1). \tag{12.3}$$

The avatars of the regular Black–Scholes formulas can be obtained as follows:

- *Stock without dividend (Black–Scholes):* the risky asset is X.
- *Stock with continuous yield $\lambda > 0$ of dividends:* the risky asset is $e^{\lambda t}X_t$ and one has to replace x_0 by $e^{-\lambda\tau}x_0$ in the right-hand sides of (12.1), (12.2) and (12.3).